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Isometric immersions of pseudo-Riemannian space forms

Chen Qing, Zuo Dafeng*, Cheng Yi

Department of Mathematics, University of Science and Technology of China, Hefei, Anhui 230026, PR China

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Abstract

In this paper we study local isometric immersions $f : M_s^n(K) \to N_{s+q}^{2n-1}(c)$ of a time-like n-submanifold $M_s^n(K)$ with constant sectional curvature K and index s into a pseudo-Riemannian space form $N_{s+q}^{2n-1}(c)$ with constant sectional curvature c and index $s+q$, where $q \geq 0, 1 \leq s \leq n-1$ and $K \neq c$. We first prove the existence of Chebyshev coordinates of a time-like submanifold $M_s^n(K)$ in certain conditions. Afterwards, we generalize the classical Bäcklund theorem for space-like (or time-like) submanifolds in $N_{n-1}^{2n-1}(c)$ and $N_1^{2n-1}(c)$. Finally as an application, in the Chebyshev coordinates, we use the Bäcklund theorem to give a Bäcklund transformation and a permutability formula between the generalized sine-Laplace equation and the generalized sinh-Laplace equation.

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1. Introduction

The classical Bäcklund theorem [2,3] studies the transformation of surfaces with constant negative curvature in Euclidean space E^3 by realizing them as the focal surfaces of a pseudo-spherical line congruence. The integrability theorem says that one can construct a new surface in E^3 with constant negative curvature from a given one by using the Bäcklund

* Corresponding author.
E-mail addresses: qchen@ustc.edu.cn (C. Qing), dfzuo@ustc.edu.cn (Z. Dafeng), ycheng@ustc.edu.cn (C. Yi).

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transformation (BT). With the development of integrable theory, BT has become an important method to find new solutions of partial differential equations. At the same time, many authors have presented some generalizations of geometric Bäcklund theorem. In [3], Chern and Terng introduced W-congruence and discussed BT between affine minimal surfaces in affine geometry. In [4–6], Tenenblat and Terng considered the generalization in higher dimensional space forms $N^{2n-1}(c)$ and obtained the generalized sine-Gordon and wave equation. On the other hand, the pseudo-Riemannian geometry has been a subject of wide interest [7]. In Lorentzian space forms $N_1^3(c)$, the generalization was considered in [8–14].

Note that the natural generalization of BT is closely related to local isometric immersions in (pseudo-Riemannian) space forms, which is a classical problem of differential geometry. Cartan showed that an n-dimensional hyperbolic space form can be locally immersed in E^{2n-1} and the dimension $2n-1$ cannot be lowered [1,17]. It is a classical result due to Hilbert [18] that there are no complete isometric immersions $M^2(K) \to N^3(c)$ if $K < c$ and $K < 0$, but it is yet unknown (though conjectured) whether this result extends to complete isometric immersions $M^n(K) \to N^{2n-1}(c)$ for $K < c$ and $K < 0$. Notice that for the case $K = 0$, one always has the clifford tori, and $K > 0$ cannot occur due to the fact that such immersions induce global Chebyshev coordinates [19,20]. In contrast, when $K > c$, one always has the totally umbilical hypersurfaces. Especially, if the immersion has no umbilic points, then the normal bundle is flat [19]. For pseudo-Riemannian space forms, there are some similar results [23,24]. For instance, in [23] the solution of the generalized equation has been shown to correspond to Riemannian submanifolds $M^n(K)$ with constant sectional curvature in pseudo-Riemannian space forms $N_q^{2n-1}(c)$ of index q, with $K \neq c$, flat normal bundle, and the principal normal curvatures are different from $K - c$. In [24] (or see [25]), Borisenko has proved if $H_s^n(-1)$ is a complete connected pseudo-Riemannian manifold with constant negative curvature and $s \neq 0, 1, 3, 7$, then the manifold $H_s^n(-1)$ cannot be isometrically immersed into E_s^{2n-1}.

The aim of this paper is to study local isometric immersions $f : M_s^n(K) \to N_{s+q}^{2n-1}(c)$ of a time-like n-submanifold $M_s^n(K)$ with constant sectional curvature K into a pseudo-Riemannian space form $N_{s+q}^{2n-1}(c)$ of index $s+q$, where $K \neq c$. In order to avoid degenerate cases we shall make the following assumptions on isometric immersions $f : M_s^n(K) \to N_{s+q}^{2n-1}(c)$:

(1) the second fundamental form of f is orthogonally diagonalizable; and
(2) there exists a point p of M where principal normal curvatures are different from $K - c$.

Based on the above assumptions, we obtain a correspondence (Theorem 3.2) between isometric immersions $f : M_s^n(K) \to N_{s+q}^{2n-1}(c) (K \neq c)$ and solutions of the generalized system are

$$\epsilon_i (f_{ij})_{x_i} + \epsilon_j (f_{ji})_{x_j} + \sum_{k=1}^{n} \epsilon_k f_{ki} f_{kj} = -K a_{ni} a_{nj} \quad \text{if } \; i \neq j,$$
$$(f_{ij})_{x_k} = f_{ik} f_{kj} \quad \text{if } \; i, j, k \text{ are distinct}, \qquad (a_{ij})_{x_k} = a_{ik} f_{kj} \quad \text{if } \; j \neq k,$$
$$AJA^t = J, \quad A = (a_{ij}), \quad J = \mathrm{diag}(J_{11}, \dots, J_{nn}). \tag{1.1}$$

When $K > c$, (1.1) is the generalized homogenous wave equation for $K = 0$, and the generalized sinh-Gordon equation for $K \neq 0$. When $K < c$, (1.1) is the generalized Laplace equation for $K = 0$, and the generalized sine-Laplace (GSL) equation for $K \neq 0$. In fact, when $q = 0$ and $K < c$, the above assumptions (1) and (2) are needless (Theorem 3.3). By using the correspondence between isometric immersions and the system (1.1), we give the higher dimensional generalizations of the classical Bäcklund theorem in R^{2n-1}_{n-1} and R^{2n-1}_{1}. As an application, by introducing the Chebyshev coordinates, we use the Bäcklund theorem to give an explicit BT and a permutability formula between the generalized sine-Laplace equation and the generalized sinh-Laplace equation (GSHL, Theorems 4.7 and 4.10).

2. Moving frames for time-like submanifolds in $N^{2n-1}_{s+q}(c)$

Let $N^{2n-1}_{s+q}(c)$ be a $(2n-1)$-dimensional pseudo-Riemannian space forms with index $s+q$ and constant sectional curvature c. We take $\{e_A | A = 1, 2, \dots, 2n-1\}$ the local pseudo-orthogonal frame of $N^{2n-1}_{s+q}(c)$, such that

$$\langle e_A, e_B\rangle = \epsilon_A \delta^A_B, \tag{2.1}$$

where $\epsilon_A = 1$ $(1 \leq A \leq n-s$ or $n+1 \leq A \leq 2n-q-1)$ and $\epsilon_A = -1$ $(n-s+1 \leq A \leq n$ or $2n-q \leq A \leq 2n-1)$. In this section, we use the following index conventions unless otherwise stated:

$$1 \leq i, j, k \leq n; \quad n+1 \leq \alpha, \beta, \gamma \leq 2n-1; \quad 1 \leq A, B, C \leq 2n-1. \tag{2.2}$$

Let $f : M^n_s \to N^{2n-1}_{s+q}(c)$ be an immersed time-like submanifold of index s. One may choose a local pseudo-orthonormal frame $\{f; e_A\}$ defined on an open domain V of M such that $\{e_i\}$ are tangent and $\{e_\alpha\}$ are normal to M, respectively. Let $\{\omega^A\}$ be the dual coframe of $\{e_A\}$ defined by $\omega^A(e_B) = \delta^A_B$. Then one can write

$$\mathrm{d}f = \sum_A \omega^A e_A, \quad \langle e_A, e_B\rangle = \epsilon_A \delta^B_A. \tag{2.3}$$

It is well known that there exist connection 1-forms $\{\omega^B_A\}$ such that structural equations of $N^{2n-1}_{s+q}(c)$ are given by

$$\mathrm{d}\omega^A = \sum_B \omega^B \wedge \omega^A_B, \qquad \mathrm{d}\omega^B_A = \sum_C \omega^C_A \wedge \omega^B_C - c\epsilon_A \omega^A \wedge \omega^B, \tag{2.4}$$

where $\epsilon_A \omega^B_A + \epsilon_B \omega^A_B = 0$. Restricting these forms to M, one has

$$\omega^\alpha = 0, \quad \mathrm{d}\omega^\alpha = \sum_i \omega^i \wedge \omega^\alpha_i. \tag{2.5}$$

By Cartan's lemma, one may set

$$\omega^\alpha_i = \sum_j h^\alpha_{ij} \omega^j, \quad h^\alpha_{ij} = h^\alpha_{ji}. \tag{2.6}$$

The first equation of (2.4) gives

$$\mathrm{d}\omega^i = \sum_j \omega^j \wedge \omega^i_j, \qquad \epsilon_i \omega^j_i + \epsilon_j \omega^i_j = 0, \tag{2.7}$$

where (ω^j_i) is the connection on M and uniquely determined by these equations.

The Gauss–Codazzi–Ricci equations are

$$\mathrm{d}\omega_i^j = \sum_k \omega_i^k \wedge \omega_k^j + \Omega_i^j, \tag{2.8}$$

$$\mathrm{d}\omega_i^\alpha = \sum_A \omega_i^A \wedge \omega_A^\alpha, \tag{2.9}$$

$$\mathrm{d}\omega_\alpha^\beta = \sum_\gamma \omega_\alpha^\gamma \wedge \omega_\gamma^\beta + \Omega_\alpha^\beta, \tag{2.10}$$

where $\Omega_i^j = \sum_\alpha \omega_i^\alpha \wedge \omega_\alpha^j - c\epsilon_i \omega^i \wedge \omega^j$ are the curvature tensors and $\Omega_\alpha^\beta = \sum_k \omega_\alpha^k \wedge \omega_k^\beta$ are the normal curvature tensors. M is said to have a constant curvature K if and only if $\Omega_i^j = -\epsilon_i K \omega^i \wedge \omega^j$.

The two fundamental forms of M are

$$\mathrm{I} = \langle \mathrm{d}f, \mathrm{d}f \rangle = \sum_i \epsilon_i (\omega^i)^2, \qquad \mathrm{II} = -\sum_\alpha \langle \mathrm{d}f, \mathrm{d}e_\alpha \rangle e_\alpha = \sum_{i,\alpha} \epsilon_\alpha \omega_i^\alpha \omega^i e_\alpha. \tag{2.11}$$

Let $\nabla^\perp$ be an induced connection of the normal bundle $\vartheta(M)$ of M, that is $\nabla^\perp e_\alpha = \omega_\alpha^\beta e_\beta$. A vector field $\eta \in \vartheta(M)$ is parallel if $\nabla^\perp \eta = 0$. The normal bundle $\vartheta(M)$ is flat if $\nabla^\perp$ is flat, that is, $\Omega_\alpha^\beta = 0$. If the normal bundle is flat, one may choose a local orthonormal frame field $\{e_\alpha\}$ for the normal bundle such that $\omega_\alpha^\beta = 0$. If there exists a pseudo-orthogonal basis $\{e_i\}$ such that $h_{ij}^\alpha = 0$ for all α when $i \neq j$, we call the second fundamental form to be orthogonally diagonalizable. Given $\zeta \in \vartheta(M)$, one may define the shape operator by

$$\langle A_\zeta(e_i), e_j \rangle = \epsilon_j \langle \mathrm{II}(e_i, e_j), \zeta \rangle, \tag{2.12}$$

that is to say, $A_{e_\alpha}(e_i) = \sum_j h_{ij}^\alpha e_j$. If $\mathrm{rank}\{A_\zeta | \zeta \in \vartheta(M)\} = \dim M$, the normal bundle is called non-degenerate. Obviously when the second fundamental form of the immersion is orthogonally diagonalizable, then the normal bundle must be flat. But conversely it is not true in general, the main reason is the family $\{A_\zeta | \zeta \in \vartheta(M)\}$ of shape operators of M at p is not a family of commuting self-adjoint operators on T_pM, hence generically there isn't a smooth common eigenframe.

If the second fundamental form of M is orthogonally diagonalizable, one can write

$$B(e_j, e_j) = \sum_\alpha \omega_j^\alpha(e_j) e_\alpha = \sum_\alpha h_{jj}^\alpha e_\alpha.$$

It follows from the Gauss equations (2.8) that

$$\langle B(e_i, e_i), B(e_j, e_j) \rangle = \sum_{l=1}^{n-1} \epsilon_{n+l} h_{ii}^{n+l} h_{jj}^{n+l} = \epsilon_i \epsilon_j (K - c), \quad i \neq j. \tag{2.13}$$

In this case the vectors $\{e_i\}$ are the principal directions of M, and the corresponding principal normal curvatures of M are given by

$$\langle B(e_i, e_i), B(e_i, e_i) \rangle = \sum_{l=1}^{n-1} \epsilon_{n+l} (h_{ii}^{n+l})^2. \tag{2.14}$$

3. Isometric immersions of pseudo-Riemannian space forms

In the rest of this paper we suppose that M is of constant curvature K and $K \neq c$. To obtain the corresponding Gauss–Codazzi–Ricci equations having specially nice forms, in the following we consider the existence of Chebyshev coordinates. Firstly, we give a theorem due to [23], which gives the existence of Chebyshev coordinates of constant curved space-like submanifolds in $N_q^{2n-1}(c)$ with $q \neq 0$ (when $q = 0$, the result is obtained in [1,17,19,21], with some different conditions).

Theorem 3.1 (Barbosa et al. [23]). *Let $f : M^n(K) \to N_q^{2n-1}(c)$ be a local isometric space-like immersion, where $0 \le q \le n-1$. Assume that the normal bundle is flat and there exists a point p of M where the principal normal curvatures are different from $K - c$. Then on an open contractible region U of p, there exist line of curvature coordinates $\{x_1, \dots, x_n\}$ such that the first and second fundamental forms are*

$$\mathrm{I} = \sum_{i=1}^{n} a_{1i}^2\,\mathrm{d}x_i^2, \qquad \mathrm{II} = \sqrt{|K-c|}\sum_{i=2,\,j=1}^{n} J_{ii}a_{ij}a_{1j}\,\mathrm{d}x_i^2 e_{n+i-1}, \tag{3.1}$$

where $\{e_\alpha\}$ are local parallel normal frame fields. The Gauss–Codazzi–Ricci equations of M are

$$\begin{aligned}
&(f_{ij})_{x_i} + (f_{ji})_{x_j} + \sum_{k=1}^{n} f_{ki}f_{kj} = -Ka_{1i}a_{1j} \quad \text{if } i \neq j,\\
&(f_{ij})_{x_k} = f_{ik}f_{kj} \quad \text{if } i, j, k \text{ are distinct}, \qquad (a_{ij})_{x_k} = a_{ik}f_{kj} \quad \text{if } j \neq k,\\
&AJA^t = J, \quad A = (a_{ij}), \quad J = \mathrm{diag}(J_{11}, \dots, J_{nn}),
\end{aligned} \tag{3.2}$$

where

$$J_{ll} = \begin{cases} 1 & 1 \le l \le n-q, \\ -1 & n-q+1 \le l \le n, \end{cases}$$

when $K < c$, and

$$J_{ll} = \begin{cases} -1 & 1 \le l \le q+1, \\ 1 & q+2 \le l \le n, \end{cases}$$

when $K > c$.

Conversely, if $A = (a_{ij})$ is a solution of (3.2) defined on a simply connected domain M such that $a_{1i} (1 \le i \le n)$ does not vanish. Then there exists a space-like immersion $f : M^n \to N_q^{2n-1}(c)$ which is unique to a rigid motion of $N_q^{2n-1}(c)$ such that the two fundamental forms are given by (3.1).

When M is a time-like constant curved submanifold, we use exactly the similar steps and arguments used in Refs. [23,26] to obtain the following result.

Theorem 3.2. *Let $f : M_s^n(K) \to N_{s+q}^{2n-1}(c)$ be a local isometric immersion, where $q \ge 0$ and $1 \le s \le n-1$. Assume that the second fundamental form is orthogonally* *diagonalizable and there exists a point p of M where the principal normal curvatures are different from $K - c$. Then on an open contractible region U of p, there exist line of curvature coordinates $\{x_1, \dots, x_n\}$ such that the first and second fundamental forms are*

$$\mathrm{I} = \sum_{i=1}^{n} \epsilon_i a_{ni}^2 \, \mathrm{d}x_i^2, \qquad \mathrm{II} = -\sqrt{|K-c|} \sum_{i=1}^{n} \sum_{l=1}^{n-1} J_{ll} \epsilon_i a_{ni} a_{li} \, \mathrm{d}x_i^2 \, e_{n+l}, \tag{3.3}$$

where $\{e_\alpha\}$ are local parallel normal frame fields. The Gauss–Codazzi–Ricci equations of M are

$$\begin{aligned} &\epsilon_i (f_{ij})_{x_i} + \epsilon_j (f_{ji})_{x_j} + \sum_{k=1}^{n} \epsilon_k f_{ki} f_{kj} = -K a_{ni} a_{nj}, \quad \text{if } i \neq j, \\ &(f_{ij})_{x_k} = f_{ik} f_{kj} \quad \text{if } i, j, k \text{ are distinct}, \qquad (a_{ij})_{x_k} = a_{ik} f_{kj} \quad \text{if } j \neq k, \\ &AJA^t = J, \quad A = (a_{ij}), \quad J = \mathrm{diag}(J_{11}, \dots, J_{nn}), \end{aligned} \tag{3.4}$$

where $J_{nn} = 1$, and $J_{ll} = \epsilon_{n+l}$ when $K < c$ and $-\epsilon_{n+l}$ when $K > c$.

Conversely, if $A = (a_{ij})$ is a solution of (3.4) defined on a simply connected domain M such that a_{ni} $(1 \leq i \leq n)$ does not vanish. Then there exists a time-like immersion $f : M_s^n \to N_{s+q}^{2n-1}(c)$ which is unique to a rigid motion of $N_{s+q}^{2n-1}(c)$ such that the two fundamental forms are given by (3.3).

Proof. Since the second fundamental form is orthogonally diagonalizable, there exist local parallel normal frame fields $\{e_\alpha\}$ and $\{e_i\} \in T_pM$ such that $\omega_\alpha^\beta = 0$ and $h_{ij}^\alpha = 0$ $(i \neq j)$. It follows from the hypothesis that there is an open subset V of M such that, at each point of V, the principal normal curvatures are different from $K - c$. Hence one may define some functions a_{ij} on V by:

$$a_{ni} = \sqrt{\frac{\lambda_i (K-c)}{\langle B(e_i, e_i), B(e_i, e_i)\rangle - K + c}}, \qquad a_{li} = -\frac{\epsilon_i a_{ni} h_{ii}^{n+l}}{\sqrt{|K-c|}}, \tag{3.5}$$

where $\lambda_i = \pm 1$ is chosen so that the right-hand side is positive. From (3.5), one has

$$\mathrm{d}\frac{1}{a_{ni}} = \frac{\epsilon_i a_{ni} \sum_{l=1}^{n-1} h_{ii}^{n+l} \, \mathrm{d}h_{ii}^{n+l}}{K-c}. \tag{3.6}$$

It follows from the Codazzi equations (2.9) that

$$\mathrm{d}h_{ii}^{n+l} \wedge \omega^i + h_{ii}^{n+l} \, \mathrm{d}\omega^i = \sum_{j=1}^{n} h_{jj}^{n+l} \omega_i^j \wedge \omega^j. \tag{3.7}$$

By using (2.13), (3.6) and (3.7), one obtains

$$\mathrm{d}\frac{\omega^i}{a_{ni}} = \lambda_i a_{ni}\left(\mathrm{d}\omega^i + \frac{\sum_{l=1}^{n-1}\sum_{j=1}^{n}\epsilon_{n+l}h_{ii}^{n+l}h_{jj}^{n+l}\omega_i^j\wedge\omega^j}{K-c}\right)$$
$$= \lambda_i a_{ni}(\mathrm{d}\omega^i - \omega^j\wedge\omega_j^i) = 0 \qquad (3.8)$$

for all $1 \le i \le n$. Hence on an open contractible region U of V, there exist smooth real valued functions $\{x_1, \dots, x_n\}$ such that

$$\omega^i = \epsilon_i a_{ni}\,\mathrm{d}x_i, \quad 1 \le i \le n. \qquad (3.9)$$

By using (2.5), one gets

$$\omega_i^j = -f_{ji}\,\mathrm{d}x_i + \epsilon_i\epsilon_j f_{ij}\,\mathrm{d}x_j, \quad f_{ij} = \frac{(a_{nj})_{x_i}}{a_{ni}}, \quad \omega_i^{n+l} = -a_{li}\sqrt{|K-c|}\,\mathrm{d}x_i. \qquad (3.10)$$

Substituting (3.9) and (3.10) into (2.8)–(2.10), one has the Gauss–Codazzi–Ricci equations (3.4). Substituting (3.5) into (2.13), one gets $\sum_{k=1}^{n} J_{kk}a_{ki}a_{kj} = 0\ (i \ne j)$ which implies A^tJA is a diagonal matrix. In the following one only need to prove $AJA^t = J$.

Let $W = \vartheta_p M \oplus \mathbf{R}$, where $\vartheta_p M$ is the normal bundle of M at p. Consider the inner product

$$\langle\langle (x,s),(y,t)\rangle\rangle = \langle x,y\rangle - (K-c)st, \quad x,y \in \vartheta_p M. \qquad (3.11)$$

Since $K \ne c$, $\langle\langle,\rangle\rangle$ is a pseudo-Riemannian product which has index q (resp. $q+1$) if $K < c$ (resp. $K > c$). Define a map $\beta : T_pM \times T_pM \to W$ by $\beta(x,y) = (B(x,y), \langle x,y\rangle)$, where $x, y \in T_pM$. Using (2.13), it is easily verified that $\langle\langle\beta(x,y),\beta(w,z)\rangle\rangle = \langle\langle\beta(x,w),\beta(y,z)\rangle\rangle$ which implies that, according to the terminology of [19], β is a flat bilinear form with respect to $\langle\langle,\rangle\rangle$, where $x, y, w, z \in T_pM$. By a direct calculation, one may know that $\{(a_{ni}/\sqrt{|K-c|})\beta(e_i,e_i)\}$ is a pseudo-orthonormal basis for W. Hence one can reorder $\{e_i\}$ such that

$$\frac{a_{ni}^2}{|K-c|}(\langle B(e_i,e_i), B(e_i,e_i)\rangle - K + c)\begin{cases} J_{ii} & \text{if } K < c, \\ -J_{ii} & \text{if } K > c. \end{cases} \qquad (3.12)$$

It follows from (3.5) that $AJA^t = J$.

The converse follows from the fundamental theorem of pseudo-Riemannian geometry [7]. This completes the proof of the theorem. □

In the case of $K < c$ and $q = 0$, we can prove that the second fundamental form of M is necessarily orthogonally diagonalizable.

Theorem 3.3. *Let $f : M_s^n(K) \to N_s^{2n-1}(c)$ be a local isometric immersion and $K < c$. Then*

(1) *the normal bundle is flat; and*

(2) *on an open contractible region U of p, there exist line of curvature coordinates* $\{x_1, \dots, x_n\}$ *such that the first and second fundamental forms are*

$$\mathrm{I} = \sum_{i=1}^{n} \epsilon_i a_{ni}^2 \, \mathrm{d}x_i^2, \qquad \mathrm{II} = -\sqrt{c-K} \sum_{i=1}^{n} \sum_{l=1}^{n-1} \epsilon_i a_{ni} a_{li} \, \mathrm{d}x_i^2 \, e_{n+l}, \tag{3.13}$$

where $\{e_\alpha\}$ *are local parallel normal frame fields. The Gauss–Codazzi–Ricci equations of M are*

$$\begin{aligned} &\epsilon_i (f_{ij})_{x_i} + \epsilon_j (f_{ji})_{x_j} + \sum_{k=1}^{n} \epsilon_k f_{ki} f_{kj} = -K a_{ni} a_{nj} \quad \text{if } i \neq j, \\ &(f_{ij})_{x_k} = f_{ik} f_{kj} \quad \text{if } i, j, k \text{ are distinct}, \qquad (a_{ij})_{x_k} = a_{ik} f_{kj}, \quad \text{if } j \neq k, \\ &AA^t = I_n. \end{aligned} \tag{3.14}$$

Proof. Let $W = \vartheta_p M \oplus \mathbf{R}$, where $\vartheta_p M$ is the normal bundle of M at p. Consider the inner product

$$\langle\langle (x, s), (y, t) \rangle\rangle = \langle x, y \rangle - (K - c) st, \quad x, y \in \vartheta_p M. \tag{3.15}$$

Define a map $\beta : T_p M \times T_p M \to W$ by $\beta(x, y) = (B(x, y), \langle x, y \rangle)$, where $x, y \in T_p M$. By using (2.13), it is easily verified that $\langle\langle \beta(x, y), \beta(w, z) \rangle\rangle = \langle\langle \beta(x, w), \beta(y, z) \rangle\rangle$ which implies that β is a Euclidean flat bilinear form with respect to $\langle\langle , \rangle\rangle$, where $x, y, w, z \in T_p M$. Since $K < c$ and the normal bundle is space-like, $\langle\langle , \rangle\rangle$ is a Riemannian product. By using Theorem 2(a) in [19], one may choose a pseudo-orthonormal basis $\{e_i\} \in T_p M$ which diagonalizes β. Hence the second fundamental form is orthogonally diagonalizable, that is, $B(e_i, e_j) = 0 \ (i \neq j)$. Then we have $h_{ij}^\alpha = 0 \ (i \neq j)$, and $\omega_\alpha^\beta = 0$ which implies the normal bundle is flat. Note that the normal bundle is space-like, then (2.13) becomes

$$\sum_{\alpha=n+1}^{2n-1} h_{ii}^\alpha h_{jj}^\alpha = \epsilon_i \epsilon_j (K - c), \quad i \neq j. \tag{3.16}$$

Since $K < c$, one may define some functions a_{ij} on V by:

$$a_{ni} = \sqrt{\frac{c-K}{\langle B(e_i, e_i), B(e_i, e_i) \rangle - K + c}}, \qquad a_{li} = -\frac{\epsilon_i a_{ni} h_{ii}^{n+l}}{\sqrt{c-K}}. \tag{3.17}$$

Substituting (3.17) into (3.16), one gets $\sum_{k=1}^{n} a_{ki} a_{kj} = 0$ and $\sum_{k=1}^{n} a_{ki}^2 = 1$ which imply $A \in \mathrm{O}(n)$, i.e., $AA^t = I_n$. From (3.17), one has

$$\mathrm{d}\frac{1}{a_{ni}} = \frac{a_{ni} \sum_{l=1}^{n-1} h_{ii}^{n+l} \, \mathrm{d}h_{ii}^{n+l}}{K - c}. \tag{3.18}$$

It follows from the Codazzi equations (2.9) that

$$\mathrm{d}h_{ii}^{n+l} \wedge \omega^i + h_{ii}^{n+l} \, \mathrm{d}\omega^i = \sum_{j=1}^{n} h_{jj}^{n+l} \omega_i^j \wedge \omega^j. \tag{3.19}$$

By using (3.16), (3.18) and (3.19), one obtains

$$\mathrm{d}\frac{\omega^i}{a_{ni}} = a_{ni}\left(\mathrm{d}\omega^i + \frac{\sum_{l=1}^{n-1}\sum_{j=1}^{n} h_{ii}^{n+l}h_{jj}^{n+l}\omega_i^j \wedge \omega^j}{K-c}\right) = a_{ni}(\mathrm{d}\omega^i - \omega^j \wedge \omega_j^i) = 0 \tag{3.20}$$

for all $1 \le i \le n$. Hence on an open contractible region U of V, there exist smooth real valued functions $\{x_1, \ldots, x_n\}$ such that

$$\omega^i = \epsilon_i a_{ni}\,\mathrm{d}x_i, \quad 1 \le i \le n. \tag{3.21}$$

By using (2.5), one gets

$$\omega_i^j = -f_{ji}\,\mathrm{d}x_i + \epsilon_i\epsilon_j f_{ij}\,\mathrm{d}x_j, \qquad f_{ij} = \frac{(a_{nj})_{x_i}}{a_{ni}}, \qquad \omega_i^{n+l} = -a_{li}\sqrt{c-K}\,\mathrm{d}x_i. \tag{3.22}$$

Substituting (3.21) and (3.22) into (2.8)–(2.10), one has the Gauss–Codazzi–Ricci equations (3.14). This completes the proof of the theorem. □

Analogous to the proof of the above theorem, we can obtain the following theorem which has been obtained in [8] for the case $K > c$ and $c = 0$.

Theorem 3.4. *Let $f : M^n(K) \to N_{n-1}^{2n-1}(c)$ be a local isometric immersion and $K > c$. Then*

(1) *the normal bundle is flat; and*
(2) *on an open contractible region U of p, there exist line of curvature coordinates $\{x_1, \ldots, x_n\}$ such that the first and second fundamental forms are*

$$\mathrm{I} = \sum_{i=1}^{n} a_{1i}^2\,\mathrm{d}x_i^2, \qquad \mathrm{II} = -\sqrt{K-c}\sum_{i=2,j=1}^{n} a_{ij}a_{1j}\,\mathrm{d}x_i^2\,e_{n+i-1}, \tag{3.23}$$

where $\{e_\alpha\}$ are local parallel normal frame fields. The Gauss–Codazzi–Ricci equations of M are

$$\begin{aligned} &(f_{ij})_{x_i} + (f_{ji})_{x_j} + \sum_{k=1}^{n} f_{ki}f_{kj} = -Ka_{1i}a_{1j} \quad \text{if } i \ne j,\\ &(f_{ij})_{x_k} = f_{ik}f_{kj} \quad \text{if } i, j, k \text{ are distinct}, \qquad (a_{ij})_{x_k} = a_{ik}f_{kj} \quad \text{if } j \ne k,\\ &AA^t = I_n, \quad A = (a_{ij}). \end{aligned} \tag{3.24}$$

Example 3.5 (Milnor [15], Gu et al. [22] and Tenenblat [26]). Consider the case $n = 2$ and $q = 1$ in Theorem 3.1. When $K < c$, by choosing

$$A = \begin{pmatrix} \cosh\dfrac{u}{2} & \sinh\dfrac{u}{2} \\ \sinh\dfrac{u}{2} & \cosh\dfrac{u}{2} \end{pmatrix},$$

where u is a differentiable function of x_1, x_2, equation (3.2) reduces to $u_{x_1x_1} + u_{x_2x_2} = -K\sinh u$ which is the sinh-Laplace equation when $K \neq 0$, and the Laplace equation when $K = 0$. When $K > c$, by choosing

$$A = \begin{pmatrix} \cos\dfrac{u}{2} & \sin\dfrac{u}{2} \\ -\sin\dfrac{u}{2} & \cos\dfrac{u}{2} \end{pmatrix},$$

where u is a differentiable function of x_1, x_2, equation (3.2) reduces to $u_{x_1x_1} - u_{x_2x_2} = -K\sin u$ which is the sine-Gordon equation when $K \neq 0$, and the homogenous wave equation when $K = 0$.

Consider the case $n = 2$, $q = 0$ and $s = 1$ in Theorem 3.2. When $K > c$, by choosing

$$A = \begin{pmatrix} \cosh\dfrac{u}{2} & \sinh\dfrac{u}{2} \\ \sinh\dfrac{u}{2} & \cosh\dfrac{u}{2} \end{pmatrix},$$

where u is a differentiable function of x_1, x_2, equation (3.4) reduces to $u_{x_1x_1} - u_{x_2x_2} = -K\sinh u$ which is the sinh-Gordon equation when $K \neq 0$, and the homogenous wave equation when $K = 0$. When $K < c$, by choosing

$$A = \begin{pmatrix} \cos\dfrac{u}{2} & \sin\dfrac{u}{2} \\ -\sin\dfrac{u}{2} & \cos\dfrac{u}{2} \end{pmatrix},$$

where u is a differentiable function of x_1, x_2, equation (3.4) reduces to $u_{x_1x_1} + u_{x_2x_2} = -K\sin u$ which is the sine-Laplace equation when $K \neq 0$, and the Laplace equation when $K = 0$.

Remark 3.6. Note that when M is time-like in $N_1^3(c)$ with $K = c + \rho^2 > 0$ ($\rho \in R$ is a constant) and imaginary principal curvatures [13–15], then there exists a local coordinate system (x, y) such that

$$\mathrm{I} = \mathrm{d}x^2 + 2\sinh\alpha\,\mathrm{d}x\,\mathrm{d}y - \mathrm{d}y^2, \qquad \mathrm{II} = 2\rho\cosh\alpha\,\mathrm{d}x\,\mathrm{d}y \tag{3.25}$$

and α satisfies the equation $\alpha_{xy} + (c + \rho^2)\cosh\alpha = 0$. This means the second fundamental form is not orthogonally diagonalizable.

4. Bäcklund theorems in R^{2n-1}_{n-1} and R^{2n-1}_1

It is well known that [8,14,22] there are three kind of line congruences in $N_1^3(c)$: space-like, time-like and light-like. Note that the “line” means geodesic of target space

$N_1^3(c)$. In general we do not consider the light-like line congruence. If there exist two focal surfaces (space-like or time-like) such that line congruences are the common tangent lines of two focal surfaces, we may separate line congruences into following cases:

(i) space-like line congruence between time-like surfaces and space-like surfaces;
(ii) space-like line congruence between space-like surfaces and space-like surfaces;
(iii) space-like line congruence between time-like surfaces and time-like surfaces;
(iv) time-like line congruence between time-like surfaces and time-like surfaces.

Furthermore, when line congruences are pseudo-spherical [16] (or [14]) line congruences in $N_1^3(c)$, then two focal surfaces have the same constant Gaussian curvature. The natural generalization would be to find a transformation theory for constant sectional curvature space-like or time-like submanifolds in a suitable pseudo-Riemannian space forms. Now we consider this question in $N_{n-1}^{2n-1}(c)$ and $N_1^{2n-1}(c)$. In this section we use the summation conventions and the following index notations unless otherwise stated:

$$2 \leq i, j, k \leq n, \quad n+1 \leq \alpha, \beta, \gamma \leq 2n-1, \quad 1 \leq A, B, C \leq 2n-1, \epsilon_A = \langle e_A, e_A \rangle. \tag{4.1}$$

4.1. Bäcklund line congruences in R_{n-1}^{2n-1} and R_1^{2n-1}

Definition 4.1. A line congruence between two n-dimensional (space-like or time-like) submanifolds M and $\tilde{M}$ in R_{n-1}^{2n-1} or R_1^{2n-1} is a diffeomorphism $\mathcal{L} : M \to \tilde{M}$ such that the line joining $p \in M$ and $\tilde{p} = \mathcal{L}(p)$ is a common tangent line for M and $\tilde{M}$.

In the following we assume that line congruences are not light-like. In general, for a line congruence $\mathcal{L} : M \to \tilde{M}$ between two n-dimensional submanifolds, the normal planes $\mathcal{V}_p$ and $\tilde{\mathcal{V}}_{\tilde{p}}$ at corresponding points p and $\tilde{p}$ are of dimension $n-1$ and both of them are perpendicular to $\overrightarrow{p\tilde{p}}$. Therefore, $\mathcal{V}_p$ and $\tilde{\mathcal{V}}_{\tilde{p}}$ lie in a $2n-2$ dimensional inner product space, there are $n-1$ angles between $\mathcal{V}_p$ and $\tilde{\mathcal{V}}_{\tilde{p}}$.

Definition 4.2. A line congruence $\mathcal{L} : M \to \tilde{M}$ between two n-dimensional (space-like or time-like) submanifolds M and $\tilde{M}$ in R_{n-1}^{2n-1} or R_1^{2n-1} is called a Bäcklund line congruence if there exist local pseudo-orthonormal frames $\{e_A\}$ and $\{\tilde{e}_A\}$ of M and $\tilde{M}$, respectively, such that

(a) $\{e_\alpha\}$ and $\{\tilde{e}_\alpha\}$ are parallel normal frames for M and $\tilde{M}$, respectively;
(b) The distance between p and $\tilde{p}$ is a positive constant r, independent of p;
(c) The $n-1$ angles between $\mathcal{V}_p$ and $\tilde{\mathcal{V}}_{\tilde{p}}$ are the same and equal to a non-zero constant τ, independent of p.

In fact, the above Bäcklund line congruence could be separated into the following four cases:

(1) anti-de Sitter line congruence $\mathcal{L}_1$: space-like line congruence between a time-like submanifold M_{n-1}^n and a space-like submanifold $\tilde{M}^n$ in R_{n-1}^{2n-1};

(2) spherical line congruence $\mathcal{L}_2$: space-like line congruence between a space-like submanifold M^n and a space-like submanifold $\tilde{M}^n$ in R^{2n-1}_{n-1};
(3) de Sitter line congruence $\mathcal{L}_3$: space-like line congruence between a time-like submanifold M^n_{n-1} and a time-like submanifold $\tilde{M}^n_{n-1}$ in R^{2n-1}_{n-1};
(4) time-like de Sitter line congruence $\mathcal{L}_4$: time-like line congruence between a time-like submanifold M^n_1 and a time-like submanifold $\tilde{M}^n_1$ in R^{2n-1}_1.

4.2. Bäcklund theorems in R^{2n-1}_{n-1} and R^{2n-1}_1

Theorem 4.3. *Let M and $\tilde{M}$ be two n-dimensional submanifolds in R^{2n-1}_{n-1} or R^{2n-1}_1. Let $\mathcal{L}_i : M \to \tilde{M}$ $(1 \le i \le 4)$ be one of the above Bäcklund congruences as in Definition 4.2. Then M and $\tilde{M}$ have the same constant sectional curvature K, where* $K = -\cosh^2\tau/r^2$ in (1), $K = \sinh^2\tau/r^2$ in (2), $K = \sinh^2\tau/r^2$ in (3) and $K = \sin^2\tau/r^2$ in (4).

Proof.

Case 1. Let M and $\tilde{M}$ be a time-like submanifold M^n_{n-1} and a space-like submanifold $\tilde{M}^n$ in R^{2n-1}_{n-1}, respectively, and $f : M \to R^{2n-1}_{n-1}$ and $\tilde{f} : \tilde{M} \to R^{2n-1}_{n-1}$. Let $\mathcal{L}_1 : M \to \tilde{M}$ be an anti-de Sitter line congruence as in the above, then there exist local pseudo-orthonormal frames $\{e_A\}$ and $\{\tilde{e}_A\}$ of M and $\tilde{M}$, respectively, such that $\{e_\alpha\}$ and $\{\tilde{e}_\alpha\}$ are parallel normal frames for M and $\tilde{M}$ respectively, and for all $x \in M$,

$$\tilde{f} = f + re_1 \tag{4.2}$$

and

$$(\tilde{e}_1, \dots, \tilde{e}_{2n-1}) = (e_1, \dots, e_{2n-1}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sinh\tau I_{n-1} & \cosh\tau I_{n-1} \\ 0 & \cosh\tau I_{n-1} & \sinh\tau I_{n-1} \end{pmatrix}, \tag{4.3}$$

where $\epsilon_1 = \epsilon_\alpha = -\epsilon_i = 1$. Since $\{e_\alpha\}$ and $\{\tilde{e}_\alpha\}$ are parallel normal frames for M and $\tilde{M}$ respectively, one has

$$\omega^{n+j-1}_{n+i-1} = \tilde{\omega}^{n+j-1}_{n+i-1} = 0. \tag{4.4}$$

Take the exterior derivative of (4.2), one gets

$$\mathrm{d}\tilde{f} = \mathrm{d}f + r\,\mathrm{d}e_1 = \omega^1 e_1 + (\omega^i + r\omega^i_1)e_i + r\omega^{n+i-1}_1 e_{n+i-1}. \tag{4.5}$$

On the other hand, letting $\{\tilde{\omega}^1, \dots, \tilde{\omega}^n\}$ be the dual coframe of $\{\tilde{e}_1, \dots, \tilde{e}_n\}$, one obtains

$$\mathrm{d}\tilde{f} = \tilde{\omega}^1\tilde{e}_1 + \tilde{\omega}^i\tilde{e}_i = \tilde{\omega}^1 e_1 + \sinh\tau\tilde{\omega}^i e_i + \cosh\tau\tilde{\omega}^i e_{n+i-1}. \tag{4.6}$$

Comparing coefficients of $\{e_A\}$ in (4.5) and (4.6), one gets

$$\tilde{\omega}^1 = \omega^1, \qquad \sinh\tau\tilde{\omega}^i = \omega^i + r\omega^i_1, \qquad \cosh\tau\tilde{\omega}^i = r\omega^{n+i-1}_1. \tag{4.7}$$

This gives

$$\omega^i + r\omega^i_1 = r\tanh\tau\omega^{n+i-1}_1. \tag{4.8}$$

Since $\langle d\tilde{e}_{n+i-1}, \tilde{e}_{n+j-1}\rangle = -\tilde{\omega}_{n+i-1}^{n+j-1} = 0$ and $\tilde{e}_{n+i-1} = \cosh\tau e_i + \sinh\tau e_{n+i-1}$, one has

$$\omega_i^j = \tanh\tau(\omega_i^{n+j-1} - \omega_{n+i-1}^j). \tag{4.9}$$

By using $\omega_{n+i-1}^{n+j-1} = 0$ and Ricci equations, one gets

$$\omega_{n+i-1}^k \wedge \omega_k^{n+j-1} + \omega_{n+i-1}^1 \wedge \omega_1^{n+j-1} = 0. \tag{4.10}$$

By using (4.8) and (4.9), one obtains

$$\tilde{\omega}_1^{n+k-1} = -\langle d\tilde{e}_1, \tilde{e}_{n+k-1}\rangle = -\frac{\cosh\tau}{r}\omega^k, \qquad \tilde{\omega}_i^{n+k-1} = -\langle d\tilde{e}_i, \tilde{e}_{n+k-1}\rangle = \omega_{n+i-1}^k. \tag{4.11}$$

Hence one has

$$\tilde{\Omega}_1^j = \tilde{\omega}_1^{n+k-1} \wedge \tilde{\omega}_{n+k-1}^j = -\frac{\cosh^2\tau}{r^2}\tilde{\omega}^1 \wedge \tilde{\omega}^j,$$
$$\tilde{\Omega}_i^j = \tilde{\omega}_i^{n+k-1} \wedge \tilde{\omega}_{n+k-1}^j = \frac{\cosh^2\tau}{r^2}\tilde{\omega}^i \wedge \tilde{\omega}^j. \tag{4.12}$$

This implies that $\tilde{M}$ has a constant negative sectional curvature $-\cosh^2\tau/r^2$. By symmetry, M has the same sectional curvature $-\cosh^2\tau/r^2$. This completes the proof of Case 1.

Analogous to Case 1, we may prove the remaining cases. Here the corresponding pseudo-orthonormal frames are as follows:

Case 2. For all $x \in M$ in R_{n-1}^{2n-1}, $\mathcal{L}_2(x) = x + re_1(x)$ and

$$(\tilde{e}_1, \dots, \tilde{e}_{2n-1}) = (e_1, \dots, e_{2n-1})\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\tau I_{n-1} & \sinh\tau I_{n-1} \\ 0 & \sinh\tau I_{n-1} & \cosh\tau I_{n-1} \end{pmatrix}, \tag{4.13}$$

where $\epsilon_1 = \epsilon_i = -\epsilon_\alpha = 1$.

Case 3. For all $x \in M$ in R_{n-1}^{2n-1}, $\mathcal{L}_3(x) = x + re_1(x)$ and

$$(\tilde{e}_1, \dots, \tilde{e}_{2n-1}) = (e_1, \dots, e_{2n-1})\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\tau I_{n-1} & \sinh\tau I_{n-1} \\ 0 & \sinh\tau I_{n-1} & \cosh\tau I_{n-1} \end{pmatrix}, \tag{4.14}$$

where $\epsilon_1 = \epsilon_\alpha = -\epsilon_i = 1$.

Case 4. For all $x \in M$ in R_1^{2n-1}, $\mathcal{L}_4(x) = x + re_n(x)$ and

$$(\tilde{e}_1, \dots, \tilde{e}_{2n-1}) = (e_1, \dots, e_{2n-1})\begin{pmatrix} \cos\tau I_{n-1} & 0 & \sin\tau I_{n-1} \\ 0 & 1 & 0 \\ -\sin\tau I_{n-1} & 0 & \cos\tau I_{n-1} \end{pmatrix}, \tag{4.15}$$

where $\epsilon_\alpha = -\epsilon_n = 1 = \epsilon_i$ for $1 \le i \le n-1$.

Remark 4.4. In [8], Huang has obtained the BT between space-like submanifolds and space-like submanifolds in R^{2n-1}_{n-1}. To keep BTs complete, we still list it here. By using the higher generalizations of line congruences in [14], similarly we may generalize the classical Bäcklund theorem for space-like (or time-like) submanifolds in $N^{2n-1}_{n-1}(c)$ and $N^{2n-1}_{1}(c)$, where c is 1 or -1. To keep this paper brief, we omit those tedious and similar results.

Next we discuss the integrability theorem. If necessary we shall assume that n-submanifolds, which we consider, have the Chebyshev coordinates and flat non-degenerate normal bundle.

Theorem 4.5. *Suppose M is a time-like n-submanifold in R^{2n-1}_{n-1} with sectional curvature $-\cosh^2\tau/r^2$, where $r > 0$ and $\tau \neq 0$ are constants. Then given a space-like unit vector $v_0 \in T_{p_0}M$, there exist a space-like n-submanifold $\tilde{M}$ with the same sectional curvature $-\cosh^2\tau/r^2$ in R^{2n-1}_{n-1} and an anti-de Sitter line congruence $\mathcal{L}_1 : M \to \tilde{M}$ such that $\mathcal{L}_1(p_0) = p_0 + rv_0$.*

Proof. Let $\mathcal{T}$ be the idea generated by the following 1-forms:

$$\alpha^i = \omega^i + r\omega^i_1 - \tanh\tau\omega^{n+i-1}_1, \qquad \beta^j_i = \omega^j_i - \tanh\tau(\omega^{n+j-1}_i - \omega^j_{n+i-1}),$$
$$\gamma^j_i = \omega^{n+j-1}_{n+i-1}. \tag{4.16}$$

Notice that

$$\begin{aligned} \mathrm{d}\alpha^i &= \mathrm{d}\omega^i + r\,\mathrm{d}\omega^i_1 - r\tanh\tau\,\mathrm{d}\omega^{n+i-1}_1 \\ &\equiv -\frac{1}{r}\omega^1\wedge\omega^i + r\Omega^i_1 + r\tanh\tau\omega^{n+k-1}_1\wedge\omega^i_k - r\tanh^2\tau\omega^{n+k-1}_1\wedge\omega^{n+i-1}_k \bmod(\mathcal{T}) \\ &= -\frac{1}{r}\omega^1\wedge\omega^i + \frac{r}{\cosh^2\tau}\Omega^i_1 \bmod(\mathcal{T}) \end{aligned} \tag{4.17}$$

and $\Omega^i_1 = (\cosh^2/r^2)\tau\omega^1\wedge\omega^i$, hence one has $\mathrm{d}\alpha^i \equiv 0 \bmod(\mathcal{T})$, that is, $\mathrm{d}\alpha^i \in \mathcal{T}$. By a similar calculation, we obtain $\mathrm{d}\beta^j_i \in \mathcal{T}$. According to Theorem 3.3, the normal bundle of M is flat, one may gets $\mathrm{d}\gamma^j_i \in \mathcal{T}$. Hence $\mathcal{T}$ is a closed differential idea, i.e., $\mathrm{d}\mathcal{T} \subseteq \mathcal{T}$. Then by the Frobenus theorem, there exists a local pseudo-orthonormal frame field $\{e_A\}$ on M near p_0 with $e_1(p_0) = v_0$ and

$$\omega^i + r\omega^i_1 = r\tanh\tau\omega^{n+i-1}_1, \qquad \omega^j_i = \tanh\tau(\omega^{n+j-1}_i - \omega^j_{n+i-1}),$$
$$\omega^{n+j-1}_{n+i-1} = 0. \tag{4.18}$$

Suppose that near p_0, M is given by a time-like immersion $f : \mathcal{D} \to R^{2n-1}_{n-1}$, where $\mathcal{D}$ is an open subset of R^{2n-1}_{n-1}. Define $\tilde{f} = f + re_1$. Next, we shall prove that $\tilde{f}$ defines a space-like n-submanifold with constant sectional curvature $-\cosh^2\tau/r^2$ in R^{2n-1}_{n-1}, and $\mathcal{L} : M \to \tilde{M}$

defines an anti-de Sitter line congruence as in Definition 4.2. Taking the differential of $\tilde{f}$ and using (4.18), one gets

$$\mathrm{d}\tilde{f} = \mathrm{d}f + r\,\mathrm{d}e_1 = \omega^1 e_1 + \frac{r}{\cosh\tau}(\sinh\tau e_i + \cosh\tau e_{n+i-1})\omega_1^{n+i-1}. \tag{4.19}$$

According to Theorem 3.3, one may choose lines of curvatures $x = \{x_1, \dots, x_n\}$ for M near p_0 with respect to the above normal frame field $\{e_\alpha\}$ such that $v_i^0 = (1/a_{1i})(\partial/\partial x_i)|_{p_0}$, where $\{a_{1i}\}$ for $1 \le i \le n$ are the coefficients of the first fundamental form. Let $v_i = (1/a_{1i})(\partial/\partial x_i)$ and θ^i $(1 \le i \le n)$ be its dual co-frame. Set $v_{n+j-1} = e_{n+j-1}$ and $\theta_A^B = \epsilon_B \langle \mathrm{d}v_A, v_B\rangle$ for $1 \le A, B \le 2n-1$. Then one has $\theta_i^{n+j-1} = -\epsilon_i (a_{ji}/a_{1i})\theta^i$. Suppose $e_1 = \sum_{i=1}^n f_i v_i$, where $e_1(p_0) = v_0$. Hence

$$\omega^1 = \sum_{i=1}^n \epsilon_i f_i \theta^i, \qquad \omega_1^{n+j-1} = -\sum_{i=1}^n \epsilon_i \frac{a_{ji}}{a_{1i}} f_i \theta^i. \tag{4.20}$$

Let $B_i = (a_{1i}/a_{ni}, \dots, a_{n-1,i}/a_{ni}, \epsilon_i)$ for $1 \le i \le n$. It follows from $A \in \mathrm{O}(n)$ that $\{B_1, \dots, B_n\}$ are mutually orthogonal. Hence $\{\omega^1, \omega_1^{n+1}, \dots, \omega_1^{2n-1}\}$ are linearly independent. This means that $\tilde{f}$ has rank n and defines a space-like n-submanifold in R_{n-1}^{2n-1}. By a similar calculation as in the above theorem, one easily knows that $\tilde{M}$ is a space-like n-submanifold with constant sectional curvature $-\cosh^2\tau/r^2$ in R_{n-1}^{2n-1}, and $\mathcal{L} : M \to \tilde{M}$ is the anti-de Sitter line congruence $\mathcal{L}_1$ as in Definition 4.2. □

Similar to Case 1, we also have the following integrability theorems to the other cases.

Theorem 4.6. *Let $r > 0$ and $\tau \ne 0$ be two constants.*

(i) *If M^n (or M_{n-1}^n) is a space-like (or time-like) n-submanifold in R_{n-1}^{2n-1} with sectional curvature $\sinh^2\tau/r^2$. Then given a space-like unit vector $v_0 \in T_{p_0}M$, there exist a space-like (or time-like) n-submanifold $\tilde{M}^n$ (or $\tilde{M}_{n-1}^n$) with the same sectional curvature $\sinh^2\tau/r^2$ in R_{n-1}^{2n-1} and a spherical (or de Sitter) line congruence $\mathcal{L}_2$ (or $\mathcal{L}_3$) : $M \to \tilde{M}$ such that $\mathcal{L}_2(p_0)$ (or $\mathcal{L}_3(p_0)$) $= p_0 + rv_0$.*

(ii) *If M is a time-like n-submanifold in R_1^{2n-1} with sectional curvature $\sin^2\tau/r^2$. Then given a time-like unit vector $v_0 \in T_{p_0}M$, there exist a time-like n-submanifold $\tilde{M}$ with the same sectional curvature $\sin^2\tau/r^2$ in R_1^{2n-1} and a time-like de Sitter line congruence $\mathcal{L}_0 : M \to \tilde{M}$ such that $\mathcal{L}_4(p_0) = p_0 + rv_0$.*

4.3. Bäcklund transformations in the Chebyshev coordinate

In this section, we shall use the Bäcklund theorem and integrability theorem to derive BTs and permutability formulas of the corresponding Gauss–Codazzi–Ricci equations. As an example, we only consider Case 1 and give a BT and a permutability formula between the generalize sine-Laplace equation and the generalize sinh-Laplace equation. The other cases are similar.

According to the proof of Theorem 4.5, actually (4.18) is the BT between time-like submanifolds and space-like submanifolds in R_{n-1}^{2n-1}. Next we give an explicit form in

Chebyshev coordinates. For simplicity, we choose $r = \cosh\tau$. Hence both M and $\tilde{M}$ have constant sectional curvature -1. To make notations clear, we first recall some results about M and $\tilde{M}$. For M, since it is a time-like n-submanifold with $K = -1$, it follows from Theorem 3.3 that, on an open contractible region U of p, there exist line of curvature coordinates $\{x_1, \ldots, x_n\}$ such that the first and second fundamental forms are

$$\mathrm{I} = \sum_{i=1}^{n} \epsilon_i a_{1i}^2 \,\mathrm{d}x_i^2, \qquad \mathrm{II} = -\sum_{i=2,\, j=1}^{n} \epsilon_j a_{ij} a_{1j} \,\mathrm{d}x_i^2\, e_{n+i-1}, \tag{4.21}$$

where $\{e_\alpha\}$ are local parallel normal frame fields. The Gauss–Codazzi–Ricci equations of M are the generalized sine-Laplace equation:

$$\begin{aligned}
&\epsilon_i (f_{ij})_{x_i} + \epsilon_j (f_{ji})_{x_j} + \sum_{k=1}^{n} \epsilon_k f_{ki} f_{kj} = a_{1i} a_{1j} \quad \text{if } i \neq j,\\
&(f_{ij})_{x_k} = f_{ik} f_{kj} \quad \text{if } i, j, k \text{ are distinct}, \qquad (a_{ij})_{x_k} = a_{ik} f_{kj} \quad \text{if } j \neq k,\\
&A = (a_{ij}) \in \mathrm{O}(n).
\end{aligned} \tag{4.22}$$

For $\tilde{M}$, since it is a space-like n-submanifold with $\tilde{K} = -1$ and if it satisfies the conditions in Theorem 3.1, then on an open contractible region U of p, there exist line of curvature coordinates $\{x_1, \ldots, x_n\}$ such that the first and second fundamental forms are

$$\tilde{\mathrm{I}} = \sum_{i=1}^{n} \tilde{a}_{1i}^2 \,\mathrm{d}x_i^2, \qquad \widetilde{\mathrm{II}} = -\sum_{i=2,\, j=1}^{n} \tilde{a}_{ij} \tilde{a}_{1j} \,\mathrm{d}x_i^2\, \tilde{e}_{n+i-1}, \tag{4.23}$$

where $\{\tilde{e}_\alpha\}$ are local parallel normal frame fields. The Gauss–Codazzi–Ricci equations of $\tilde{M}$ are the generalized sinh-Laplace equation:

$$\begin{aligned}
&(\tilde{f}_{ij})_{x_i} + (\tilde{f}_{ji})_{x_j} + \sum_{k=1}^{n} \tilde{f}_{ki} \tilde{f}_{kj} = \tilde{a}_{1i} \tilde{a}_{1j} \quad \text{if } i \neq j,\\
&(\tilde{f}_{ij})_{x_k} = \tilde{f}_{ik} \tilde{f}_{kj} \quad \text{if } i, j, k \text{ are distinct},\\
&(\tilde{a}_{ij})_{x_k} = \tilde{a}_{ik} \tilde{f}_{kj} \quad \text{if } j \neq k, \qquad \tilde{A} = (\tilde{a}_{ij}) \in \mathrm{O}(1, n-1).
\end{aligned} \tag{4.24}$$

Theorem 4.7. *Let $\mathcal{L}_1 : M \to \tilde{M}$ be an anti-de Sitter line congruence as in Definition 4.2. Then*

(i) *the Chebyshev coordinates of M and $\tilde{M}$ correspond under $\mathcal{L}_1$; and*
(ii) *the corresponding BT between the GSL equation (4.22) and the GSHL equation (4.24) is*

$$\mathrm{d}\tilde{A} + \tilde{A}(F\delta - J\delta F^t J) = \tilde{A} J \delta A^t D J \tilde{A} - D A \delta, \tag{4.25}$$

where $F = (f_{ij})$ with $f_{ii} = 0$ for $1 \leq i \leq n$, $\delta = \mathrm{diag}(\mathrm{d}x_1, \ldots, \mathrm{d}x_n)$ and $D = \mathrm{diag}(1/\cosh\tau,\ \tanh\tau, \ldots,\ \tanh\tau)$.

Proof. By using the same notations as in the proof of Theorem 4.5. Determine an $O(1, n-1)$-map $\Gamma = (\chi_{ij})$ by

$$e_i = \sum_{j=1}^{n} \chi_{ij} v_j, \quad 1 \le i \le n. \tag{4.26}$$

Firstly, we prove that $\Gamma = \tilde{A}$. Notice that

$$\omega^i = \sum_{j=1}^{n} \epsilon_i \epsilon_j \chi_{ij} \theta^j = \sum_{j=1}^{n} \epsilon_i \epsilon_j \chi_{ij} a_{1j}\, \mathrm{d}x_j,$$

$$\omega_i^{n+j-1} = \langle \mathrm{d}e_i, e_{n+j-1} \rangle = \sum_{k=1}^{n} \epsilon_k \chi_{ik} a_{jk}\, \mathrm{d}x_k, \tag{4.27}$$

where $1 \le i \le n$. By using (4.7), (4.11) and $A \in O(n)$, one gets

$$\begin{aligned}
\tilde{\mathrm{I}} &= (\tilde{\omega}^1)^2 + \sum_{j=2}^{n} (\tilde{\omega}^j)^2 = (\omega^1)^2 + \sum_{j=2}^{n} (\omega_1^{n+j-1})^2 \\
&= \sum_{k,i=1}^{n} \epsilon_i \epsilon_k \chi_{1k} \chi_{1i} a_{1k} a_{1i}\, \mathrm{d}x_i\, \mathrm{d}x_k + \sum_{k,i=1}^{n} \epsilon_i \epsilon_k \chi_{1k} \chi_{1i} \sum_{j=2}^{n} a_{jk} a_{ji}\, \mathrm{d}x_i\, \mathrm{d}x_k = \sum_{k=1}^{n} \chi_{1k}^2\, \mathrm{d}x_k^2, \\
\langle \widetilde{\mathrm{II}}, \tilde{e}_{n+k-1} \rangle &= \sum_{j=1}^{n} \tilde{\omega}^j \tilde{\omega}_j^{n+k-1} = -\omega^1 \omega^k + \sum_{j=2}^{n} \omega_1^{n+j-1} \omega_k^{n+j-1} \\
&= \sum_{l,i=1}^{n} \epsilon_i \epsilon_l \chi_{1l} \chi_{ki} a_{1l} a_{1i}\, \mathrm{d}x_i\, \mathrm{d}x_k + \sum_{l,i=1}^{n} \epsilon_i \epsilon_l \chi_{1l} \chi_{ki} \sum_{j=2}^{n} a_{jl} a_{ji}\, \mathrm{d}x_i\, \mathrm{d}x_l \\
&= \sum_{l=1}^{n} \chi_{1l} \chi_{kl}\, \mathrm{d}x_l^2.
\end{aligned} \tag{4.28}$$

Then comparing (4.23) and (4.28), one knows that $\{x_1, \ldots, x_n\}$ are the Chebyshev coordinates of $\tilde{M}$ and $\tilde{A} = \Gamma$. Hence the Chebyshev coordinates of M and $\tilde{M}$ correspond under $\mathcal{L}_1$.

Next we compute the BT between the GSL equation (4.22) and the GSHL equation (4.24). Note that when $r = \cosh \tau$, (4.18) becomes

$$\omega^i + \cosh \tau \omega_1^i = \sinh \tau \omega_1^{n+i-1}, \qquad \omega_i^j = \tanh \tau (\omega_i^{n+j-1} - \omega_{n+i-1}^j). \tag{4.29}$$

Write $\Omega = (\omega_i^j)$, $D = \mathrm{diag}(1/\cosh \tau, \tanh \tau, \ldots, \tanh \tau)$ and

$$W = \begin{pmatrix} \omega^1 & \omega_{n+1}^1 & \cdots & \omega_{2n-1}^1 \\ \vdots & \vdots & \ddots & \vdots \\ \omega^n & \omega_{n+1}^n & \cdots & \omega_{2n-1}^n \end{pmatrix}.$$

Then (4.29) rewrites as

$$\Omega = JWDJ - DW^t. \tag{4.30}$$

Write $\Theta = (\theta_i^j)$ and using $e_i = \sum_{j=1}^n \tilde{a}_{ij} v_j$, one gets

$$\Omega = \tilde{A}\Theta J\tilde{A}^t J + \mathrm{d}\tilde{A}J\tilde{A}^t J. \tag{4.31}$$

Note that $\Theta = F\delta - J\delta F^t J$ and $W = J\tilde{A}J\delta A^t$, where $F = (f_{ij})$, $f_{ii} = 0$ and $\delta = \mathrm{diag}(\mathrm{d}x_1, \dots, \mathrm{d}x_n)$. It follows from (4.30) and (4.31) that one could obtain (4.25). This completes the proof of the theorem. □

Remark 4.8. By analogy with the above discussion, one may obtain the BT between the space-like n-submanifold $\tilde{M}$ and the time-like n-submanifold M. The corresponding BT between the GSHL equation (4.24) and the GSL equation (4.24) is as follows

$$\tilde{\Omega} = \mathrm{d}AA^{-1} + A\tilde{\Theta}A^{-1} = \tilde{W}D - D\tilde{W}, \quad \tilde{\Theta} = \tilde{F}\delta - \delta\tilde{F}^t, \quad \tilde{W} = A\delta\tilde{A}^t, \tag{4.32}$$

where we call $\mathcal{L}_1^{-1} : \tilde{M} \to M$ to be an inverse anti-de Sitter line congruence.

Remark 4.9. Consider the case $n = 2$, we choose

$$A = \begin{pmatrix} \cos\dfrac{u}{2} & \sin\dfrac{u}{2} \\ -\sin\dfrac{u}{2} & \cosh\dfrac{u}{2} \end{pmatrix}, \qquad \tilde{A} = \begin{pmatrix} \cosh\dfrac{\tilde{u}}{2} & \sinh\dfrac{\tilde{u}}{2} \\ \sinh\dfrac{\tilde{u}}{2} & \cosh\dfrac{\tilde{u}}{2} \end{pmatrix}.$$

Then (4.25) gives the BT between the sine-Laplace equation $\Delta u = \sin u$ and the sinh-Laplace equation $\Delta\tilde{u} = \sinh\tilde{u}$ [8,11,16]:

$$\frac{1}{2}(u_{x_1} - \tilde{u}_{x_2}) = -\frac{1}{\cosh\tau}\cos\frac{u}{2}\sinh\frac{\tilde{u}}{2} - \tanh\tau\sin\frac{u}{2}\cosh\frac{\tilde{u}}{2},$$
$$\frac{1}{2}(u_{x_2} + \tilde{u}_{x_1}) = -\frac{1}{\cosh\tau}\sin\frac{u}{2}\cosh\frac{\tilde{u}}{2} + \tanh\tau\cos\frac{u}{2}\sinh\frac{\tilde{u}}{2}.$$

Finally we consider the generalization of Bianchi permutability theorem and give a permutability formula.

Theorem 4.10. *Let $\mathcal{L}_1^1 : M_0 \to M_1$ and $\mathcal{L}_1^2 : M_0 \to M_2$ be two anti-de Sitter line congruences with angles τ_1, τ_2 and distances $\cosh\tau_1$, $\cosh\tau_2$, respectively, as in Definition 4.2. If $\tau_1 \neq \tau_2$, then there exist a unique time-like n-submanifold M_3 in R_{n-1}^{2n-1} and two inverse anti-de Sitter line congruences $\tilde{\mathcal{L}}_1^1 : M_1 \to M_3$, $\tilde{\mathcal{L}}_1^2 : M_2 \to M_3$ with angles τ_2, τ_1 and distances $\cosh\tau_2$, $\cosh\tau_1$, respectively, such that $\tilde{\mathcal{L}}_1^2 \circ \mathcal{L}_1^2 = \tilde{\mathcal{L}}_1^1 \circ \mathcal{L}_1^1$. The corresponding permutability formula is*

$$A_3A_0^{-1}(D_1 - D_2A_2A_1^{-1}) = D_1A_2A_1^{-1} - D_2, \tag{4.33}$$

where $A_0, A_3 \in \mathrm{O}(n)$, $A_1, A_2 \in \mathrm{O}(1, n-1)$ and $D_l = \mathrm{diag}(1/\cosh\tau_l, \tanh\tau_l, \dots, \tanh\tau_l)$ for $l = 1, 2$.

Proof. Firstly, suppose the existence of M_3, $\tilde{\mathcal{L}}_1^1$ and $\tilde{\mathcal{L}}_1^2$, we prove the uniqueness. Let $p_0 \in M_0$, then $p_3 = \tilde{\mathcal{L}}_1^1(p_1) = \tilde{\mathcal{L}}_1^2(p_2)$. Since $\mathcal{L}_1^l$, $(\tilde{\mathcal{L}}_1^l)^{-1}$ for $l = 1, 2$ are anti-de Sitter, inverse anti-de Sitter line congruences, one has $\overrightarrow{p_0p_1}$, $\overrightarrow{p_1p_3} \in T_{p_1}M_1$ and $\overrightarrow{p_0p_2}$, $\overrightarrow{p_2p_3} \in T_{p_2}M_2$. Therefore, $\overrightarrow{p_0p_3} \in T_{p_1}M_1 \cap T_{p_2}M_2$. Note that $\tau_1 \neq \tau_2$, $T_{p_1}M_1$ and $T_{p_2}M_2$ are two n-planes in general position in R_{n-1}^{2n-1}, so $\dim T_{p_1}M_1 \cap T_{p_2}M_2 = 1$. Hence M_3 is unique determined by $\mathcal{L}_1^1$ and $\mathcal{L}_1^2$.

Secondly, we still suppose the existence of M_3, $\tilde{\mathcal{L}}_1^1$ and $\tilde{\mathcal{L}}_1^2$, we prove the permutability formula (4.33). Let $\{v_A^0\}$ be the frame field of M_0, where $\{v_i^0\}_{i=1}^n$ are the principal curvature directions and $\{v_\alpha^0\}$ are local parallel frame fields (see Theorem 4.5 or Theorem 4.7). Let $\{e_i^k, \tilde{e}_i^k\}_{i=1}^n \in T_{p_k}M_k$ and $\{v_\alpha^k\} \in \vartheta_{p_k}(M)$ $(k = 0, 1, 2)$ such that

$$\begin{aligned} &\overrightarrow{p_0p_1} = \cosh\tau_1 e_1^0, \quad e_1^1 = e_1^0, \qquad e_1^2 = \tilde{e}_1^0, \quad \overrightarrow{p_0p_2} = \cosh\tau_2 \tilde{e}_1^0;\\ &e_i^1 = \sinh\tau_1 e_i^0 + \cosh\tau_1 v_{n+i-1}^0, \qquad e_i^2 = \sinh\tau_2 \tilde{e}_i^0 + \cosh\tau_2 v_{n+i-1}^0;\\ &v_{n+i-1}^1 = \cosh\tau_1 e_i^0 + \sinh\tau_1 v_{n+i-1}^0, \qquad v_{n+i-1}^2 = \cosh\tau_2 \tilde{e}_i^0 + \sinh\tau_2 v_{n+i-1}^0. \end{aligned} \tag{4.34}$$

On the other hand,

$$\begin{aligned} &\overrightarrow{p_1p_3} = \cosh\tau_2 \tilde{e}_1^1, \qquad \overrightarrow{p_2p_3} = \cosh\tau_1 \tilde{e}_1^2;\\ &v_{n+i-1}^3 = \cosh\tau_2 \tilde{e}_i^1 + \sinh\tau_2 v_{n+i-1}^1 = \cosh\tau_1 \tilde{e}_i^2 + \sinh\tau_1 v_{n+i-1}^2. \end{aligned} \tag{4.35}$$

Let $A_k = (a_{ij}^k)$ be the corresponding map associated to M_k with respect to the normal frame field $\{v_\alpha^k\}$ for $k = 0, 1, 2, 3$ respectively, where $A_0 \in \mathrm{O}(n)$ and $A_1, A_2 \in \mathrm{O}(1, n-1)$. Then one has

$$e_i^0 = \sum_{j=1}^n a_{ij}^1 v_j^0, \qquad \tilde{e}_i^0 = \sum_{j=1}^n a_{ij}^2 v_j^0, \quad 1 \le i \le n. \tag{4.36}$$

Hence one gets

$$\tilde{e}_i^0 = \sum_{j=1}^n C_{ij} e_j^0, \quad C = (C_{ij}) = A_2 A_1^{-1}, \;\; 1 \le i \le n. \tag{4.37}$$

Similarly,

$$\tilde{e}_i^1 = \sum_{j=1}^n X_{ij} e_j^1, \qquad \tilde{e}_i^2 = \sum_{j=1}^n X_{ij} e_j^2, \quad X = A_3 A_0^{-1}, \;\; 1 \le i \le n. \tag{4.38}$$

By using $\overrightarrow{p_0p_1} + \overrightarrow{p_1p_3} = \overrightarrow{p_0p_2} + \overrightarrow{p_2p_3}$, one obtains

$$\cosh\tau_1 e_1^0 + \cosh\tau_2 \tilde{e}_1^1 = \cosh\tau_1 \tilde{e}_1^0 + \cosh\tau_2 \tilde{e}_1^2. \tag{4.39}$$

Expanding (4.39) and comparing the coefficients, one has

$$\begin{aligned} e_1^0 :\ &\cosh\tau_1 + X_{11}\cosh\tau_2\\ &= C_{11}\cosh\tau_2 + X_{11}C_{11}\cosh\tau_1 + \cosh\tau_1 \sinh\tau_2 \sum_{k=2}^n X_{1k}C_{k1}, \end{aligned}$$

$$e_j^0: \cosh\tau_2\sinh\tau_1 X_{1j} = C_{1j}\cosh\tau_2 + X_{11}C_{1j}\cosh\tau_1 + \cosh\tau_1\sinh\tau_2\sum_{k=2}^{n} X_{1k}C_{kj},$$

$$v_{n+j-1}: \cosh\tau_2\cosh\tau_1 X_{1j}\cosh\tau_2\cosh\tau_1 X_{1j}. \qquad (4.40)$$

Similarly, expanding (4.35) and comparing the coefficients, one gets that

$$e_1^0: \cosh\tau_2 X_{i1} - \cosh\tau_1 C_{11}X_{i1} - \cosh\tau_1\sinh\tau_2\sum_{k=2}^{n} X_{ik}C_{k1} = \sinh\tau_1\cosh\tau_2 C_{i1},$$

$$e_j^0: \cosh\tau_2\sinh\tau_1 X_{ij} - C_{1j}X_{i1}\cosh\tau_1 - \cosh\tau_1\sinh\tau_2\sum_{k=2}^{n} X_{ik}C_{kj}$$
$$= \sinh\tau_1\cosh\tau_2 C_{ij} - \delta_{ij}\cosh\tau_1\sinh\tau_2. \qquad (4.41)$$

Write (4.40) and (4.41) in matrix form, one obtains

$$X(D_1 - D_2C) = D_1C - D_2, \qquad D_l = \mathrm{diag}\left(\frac{1}{\cosh\tau_l}, \tanh\tau_l, \ldots, \tanh\tau_l\right) \qquad (4.42)$$

for $l = 1, 2$. Notice that $X = A_3A_0^{-1}$, hence (4.42) is (4.33). By a direct verification, one has $X = (D_1C - D_2)(D_1 - D_2C)^{-1} \in \mathrm{O}(n)$, hence $A_3 \in \mathrm{O}(n)$ (since $A_0 \in \mathrm{O}(n)$).

Finally we prove the existence, i.e., we need to prove that $\tilde{A} = A_3$ satisfies the BT (4.32) for both $A = A_1$, $\tau = \tau_2$ and $A = A_2$, $\tau = \tau_1$. By symmetry it suffices to prove $\tilde{A} = A_3$ satisfies the BT (4.32) for $A = A_1$, $\tau = \tau_2$.

Let $\{\omega^{i(k)}\}_{i=1}^n$ be the dual coframe of $\{e_i^{(k)}\}_{i=1}^n$, where $\omega_A^{B(k)}$ the corresponding for $k = 0, 1, 2$. By using (4.29), one has

$$\omega^{1(1)} = \omega^{1(0)}, \qquad \omega^{i(1)} = \omega_1^{n+i-1(0)}, \qquad \omega_i^{j(1)} = \omega_i^{j(0)}, \qquad \omega_1^{n+i-1(1)} = -\omega^{i(0)},$$
$$\omega_1^{i(1)} = \tanh\tau_1\omega^{i(0)} + \frac{1}{\cosh\tau_1}\omega_1^{n+i-1(0)}, \qquad \omega_i^{n+j-1(1)} = \omega_{n+i-1}^{j(0)}. \qquad (4.43)$$

Then

$$W^{(1)} = JW^tJ, \qquad \Omega^{(1)} = \Omega + \Lambda, \quad \Omega^{(1)} = (\omega_i^{j(1)}),$$
$$\Lambda = (\lambda_{ij}), \quad \lambda_{ij} = 0 \ \ (2 \le i, j \le n),$$
$$\lambda_{1j} = \left(\frac{1}{\cosh\tau_1} - \tanh\tau_1\right)\omega_1^{n+j-1(0)} + \left(\frac{1}{\cosh\tau_1} + \tanh\tau_1\right)\omega^{j(0)},$$
$$\lambda_{j1} = -\left(\frac{1}{\cosh\tau_1} + \tanh\tau_1\right)\omega_1^{n+j-1(0)} + \left(\frac{1}{\cosh\tau_1} - \tanh\tau_1\right)\omega^{j(0)}, \qquad (4.44)$$

where

$$W^{(1)} = \begin{pmatrix} \omega^{1(1)} & \omega_{n+1}^{1(1)} & \cdots & \omega_{2n-1}^{1(1)} \\ \vdots & \vdots & \ddots & \vdots \\ \omega^{n(1)} & \omega_{n+1}^{n(1)} & \cdots & \omega_{2n-1}^{n(1)} \end{pmatrix}.$$

It follows from $\mathcal{L}_1^1 : M_0 \to M_1$ that $\Omega = JWD_1J - D_1W^t$. By using $\tilde{e}_i^0 = \sum_{j=1}^n C_{ij}e_j^0$ and $\mathcal{L}_1^2 : M_0 \to M_2$, one gets

$$(\tilde{\Omega} =)\, \mathrm{d}CC^{-1} + C\Omega C^{-1} = J\tilde{W}D_2J - D_2\tilde{W}^t, \quad \tilde{W} = JCJW, \quad C = A_2A_1^{-1},$$
$$\mathrm{d}C = CD_1W^t - CJWJY - D_2W^t. \tag{4.45}$$

It follows from $\tilde{\mathcal{L}}_1^1 : M_1 \to M_3$ that $\Omega^{(1)} = W^{(1)}D_1 - D_1W^{(1)t}$. By using $\tilde{e}_i^1 = \sum_{j=1}^n X_{ij}e_j^1$, one knows that it suffices to prove

$$\mathrm{d}XX^{-1} + X\Omega^{(1)}X^{-1} = XW^{(1)}D_2 - D_2W^{(1)t}X^{-1}, \tag{4.46}$$

where $X = ZY^{-1}$, $Z = D_1C - D_2$ and $Y = D_1 - D_2C$. Since (4.44), (4.46) is equivalent that

$$H := \mathrm{d}XX^{-1} + X(\Omega + \Lambda)X^{-1} - XJW^tJD_2 + D_2JWJX^{-1} = 0. \tag{4.47}$$

In the following we prove (4.47). Differentiating $X = ZY^{-1}$, one gets $\mathrm{d}XX^{-1} = (XD_2 + D_1)\,\mathrm{d}CZ^{-1}$. Substituting $\mathrm{d}C$ in (4.45) into the above, one obtains

$$\begin{aligned}\mathrm{d}XX^{-1} &= (XD_2 + D_1)C(-JWJX^{-1} + D_1W^tZ^{-1}) - (XD_2 + D_1)D_2W^tZ^{-1}\\ &= (XD_1 + D_2)(-JWJX^{-1} + D_1W^tZ^{-1}) - (XD_2 + D_1)D_2W^tZ^{-1}\\ &= -XD_1JWJX^{-1} + X(D_1^2 - D_2^2)W^tZ^{-1} - D_2JWJX^{-1}.\end{aligned} \tag{4.48}$$

By using (4.48) and $\Omega = JWD_1J - D_1W^t$, one has

$$\begin{aligned}X^{-1}HZ &= VY + (D_1^2 - D_2^2)W^t - D_1W^tY - JW^tJD_2Z\\ &= [VD_1 + (D_1^2 - D_2^2)W^t - D_1W^tD_1 + JW^tJD_2^2]\\ &\quad - [VD_2 - D_1W^tD_2 + JW^tJD_2D_1]C = 0 + 0 \times C = 0,\end{aligned} \tag{4.49}$$

where $V = -D_1JWJ + JWD_1J + \Lambda$. This completes the proof of the theorem. □

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References

[1] E. Cartan, Sur les varietes de courbure constante *d*'un espace Euclidean ou non-Euclidean, Bull. Soc. Math., France 47 (1919) 125–160;
E. Cartan, Sur les varietes de courbure constante *d*'un espace Euclidean ou non-Euclidean, Bull. Soc. Math., France 48 (1920) 132–208.

[2] L.P. Eisenhart, A Treatise in the Differential Geometry of Curves and Surfaces, Ginn and Company, New York, 1909.

[3] S.-S.Chern, C.-L.Terng, An analogue of Bäcklund's theorem in affine goemetry, Rocky Mountain J. Math. 10 (1980) 105–124.

[4] K. Tenenblat, C.-L. Terng, Bäcklund's theorem for n-dimensional submanifolds in R^{2n-1}, Ann. Math. 111 (1980) 477–490.
[5] C.-L. Terng, A higher dimension generalization of the Sine-Gordon equation and its soliton theory, Ann. Math. 111 (1980) 491–510.
[6] K. Tenenblat, Bäcklund theorems for submanifolds of space forms and a generalized wave equation, Boll. Soc. Brasil. Mat. 16 (1985) 67–92.
[7] B. O'Neil, Semi-Riemannian Geometry with Applications to Gerneal Relativity, Academic Press, New York, London, 1983.
[8] Y.-Z. Huang, Bäcklund theorems in 3-dimensional Minkowski space and their high dimensional generalization, Acta Math. Sinica 24 (1986) 684–690 (in Chinese).
[9] B. Palmer, Bäcklund transformations for surfaces in Minkowski space, J. Math. Phys. 31 (1990) 2872–2875.
[10] S.G. Buyske, Bäcklund transformations of linear weingarten surfaces in Minkowski three space, J. Math. Phys. 35 (1994) 4719–4724.
[11] C. Tian, Bäcklund transformations for surfaces with $K = -1$ in $R^{2,1}$, J. Geom. Phys. 22 (1997) 212–218.
[12] J. Inoguchi, Darboux transformations on time-like constant mean curvature surfaces, J. Geom. Phys. 32 (1999) 57–78.
[13] C.-H. Gu, H.-S. Hu, J. Inoguchi, On time-like surfaces of positive constant Gaussian curvature and imaginary principal curvatures, J. Geom. Phys. 41 (2002) 296–311.
[14] D. Zuo, Q. Chen, Y. Cheng, Bäcklund theorems in 3-dimensional de Sitter Space and anti-de Sitter space, J. Geom. Phys. 44 (2–3) (2002) 279–298.
[15] T.K. Milnor, Harmonic maps and classical surface theory in Minkowski 3-space, Trans. Am. Soc. 1 (1983) 173–185.
[16] H.S. Hu, Darboux transformations between $\Delta\alpha = \sin\alpha$ and $\Delta\alpha = \sinh\alpha$ and the application to pseudo-spherical congruence, Lett. Math. Phys. 2 (1999) 187–195.
[17] J.D. Moore, Isometric immersions of space forms in space forms, Pacific J. Math. 40 (1972) 157–166.
[18] D. Hilbert, Über Flächen Von Konstanter Gausscher Krümmung, Trans. Am. Math. Soc. 2 (1901) 87–99.
[19] J.D. Moore, Submanifold of constant positive curvature I, Duke Math. J. 44 (1977) 449–484.
[20] F. Pedit, A non-immersion theorem for space forms, Comment. Math. Hekvetici 63 (1988) 672–674.
[21] M. Dajczer, R. Tojeiro, Isometric immersions and the generalized Laplace and elliptic sinh-Gordon equations, J. Reine. Angew. Math. 467 (1995) 109–147.
[22] C.-H. Gu, H.-S. Hu, Z.-X. Zhou, Darboux transformation in soliton thoery and its geometric applications, Shanhai Scientific and Technical Publishers, 1999.
[23] J.L. Barbosa, W. Ferrira, K. Tenenblat, Submanifolds of constant sectional curvature in pseudo-Riemannian manifolds, Ann. Global Anal. Geom. 14 (1996) 381–401.
[24] A.A. Borisenko, The isometric immersion of pseudo-Riemannian space of constant curvature, Ukrain. Geom. Sb. 19 (1976) 11–18 (in Russian).
[25] A.A. Borisenko, Isometric immersions of space forms into Riemannian and pseudo-Riemannian space of constant curvature, Uspekhi Mat. Nauk 56 (3) (2001) 3–78;
A.A. Borisenko, Isometric immersions of space forms into Riemannian and pseudo-Riemannian space of constant curvature, Russ. Math. Surv. 56 (3) (2001) 425–497.
[26] K. Tenenblat, Transformations of Submanifolds an Applications to Differential Equations, Addison Wesley, Longman, Pitman Monographs and Surveys in Pure and Applied Mathematics 93, 1998.